A CONSTRUCTION OF HORIKAWA SURFACE VIA $\mathbb{Q}$-GORENSTEIN SMOOTHINGS

YONGNAM LEE AND JONGIL PARK

Abstract. In this article we prove that Fintushel-Stern's construction of Horikawa surface, which is obtained from an elliptic surface via a rational blow-down surgery in smooth category, can be performed in complex category. The main technique involved is $\mathbb{Q}$-Gorenstein smoothings.

1. Introduction

As an application of a rational blow-down surgery on 4-manifolds, R. Fintushel and R. Stern showed that Horikawa surface $H(n)$ can be obtained from an elliptic surface $E(n)$ via a rational blow-down surgery in smooth category [3]. Note that Horikawa surface $H(n)$ is defined as a double cover of a Hirzebruch surface $\mathbb{F}_{n-3}$ branched over $|6C_0 + (4n-8)f|$, where C_0 is a negative section and f is a fiber of $\mathbb{F}_{n-3}$.

In this article we show that a rational blow-down surgery to obtain Horikawa surface can be performed in fact in complex category. That is, we reinterpret algebraically Fintushel-Stern's topological construction [3] of Horikawa surface $H(n)$ to give a complex structure on it. The main technique we use in this paper is $\mathbb{Q}$-Gorenstein smoothings. Note that $\mathbb{Q}$-Gorenstein smoothing theory developed in deformation theory in last thirty years is a very powerful tool to construct a non-singular surface of general type. The basic scheme is the following: Suppose that a projective surface contains several disjoint chains of curves representing the resolution graphs of special quotient singularities. Then, by contracting these chains of curves, we get a singular surface X with special quotient singularities. And then we investigate the existence of a $\mathbb{Q}$-Gorenstein smoothing of X. It is known that the cohomology $H^2(T^0_X)$ contains the obstruction space of a $\mathbb{Q}$-Gorenstein smoothing of X. That is, if $H^2(T^0_X) = 0$, then there is a $\mathbb{Q}$-Gorenstein smoothing of X. For example, we recently constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by proving the cohomology $H^2(T^0_X) = 0$ [9]. But, in general, the cohomology $H^2(T^0_X)$ is not zero and it is a very difficult problem to determine whether there exists a $\mathbb{Q}$-Gorenstein smoothing of X. In this article we also give a family of examples which admit $\mathbb{Q}$-Gorenstein smoothings even though the cohomology $H^2(T^0_X)$ does not vanish. Our main technique is a $\mathbb{Q}$-Gorenstein smoothing theory with a cyclic group action. It is briefly reviewed and developed in Section 2.

The sketch of our construction whose details are given in Section 3 is as follows: We first construct a simply connected relatively minimal elliptic surface $E(n)$ $(n \geq 5)$ with

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a special fiber, which contains two linear chains of configurations of $\mathbb{P}^1$'s

$$\underset{U_{n-3}}{\overset{-n}{\circ}} - \underset{U_{n-4}}{\overset{-2}{\circ}} - \underset{U_{n-5}}{\overset{-2}{\circ}} - \cdots - \underset{U_1}{\overset{-2}{\circ}}.$$

We construct this kind of an elliptic surface $E(n)$ explicitly by using a double cover of a blowing-up of Hirzebruch surface $\mathbb{F}_n$ branched over a special curve. The double cover of $\mathbb{F}_n$ has two rational double points A_1 and A_{2n-9}. Then its minimal resolution is an elliptic surface $E(n)$ which has an I_{2n-6} as a special fiber. Now we contract these two linear chains of configurations of $\mathbb{P}^1$'s to produce a normal projective surface X_n with two special quotient singularities, both singularities are of type $\frac{1}{(n-2)^2}(1, n-3)$. Finally, we apply $\mathbb{Q}$-Gorenstein smoothing theory with a cyclic group action developed in Section 2 for X_n in order to get our main result which is following.

Theorem 1.1. *The projective surface X_n obtained by contracting two disjoint configurations C_{n-2} from an elliptic surface $E(n)$ admits a $\mathbb{Q}$-Gorenstein smoothing of two quotient singularities all together, and a general fiber of the $\mathbb{Q}$-Gorenstein smoothing is Horikawa surface $H(n)$.*

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2. $\mathbb{Q}$-Gorenstein smoothing

In this section we briefly review a theory of $\mathbb{Q}$-Gorenstein smoothing for projective surfaces with special quotient singularities, which is a key technical ingredient in our main construction.

Definition. Let X be a normal projective surface with quotient singularities. Let $\mathcal{X} \to \Delta$ (or $\mathcal{X}/\Delta$) be a flat family of projective surfaces over a small disk Δ. The one-parameter family of surfaces $\mathcal{X} \to \Delta$ is called a $\mathbb{Q}$-*Gorenstein smoothing* of X if it satisfies the following three conditions;
(i) the general fiber X_t is a smooth projective surface,
(ii) the central fiber X_0 is X,
(iii) the canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$-Cartier.

A $\mathbb{Q}$-Gorenstein smoothing for a germ of a quotient singularity $(X_0, 0)$ is defined similarly. A quotient singularity which admits a $\mathbb{Q}$-Gorenstein smoothing is called a *singularity of class T*.

Proposition 2.1 ([8, 12, 16])**.** *Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$ for some integers a, n, d with a and n relatively prime.*

Proposition 2.2 ([8, 12, 17]). (1) *The singularities* $\overset{-4}{\circ}$ *and* $\overset{-3}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\cdots-\overset{-2}{\circ}-\overset{-3}{\circ}$ *are of class* T*.*

(2) *If the singularity* $\overset{-b_1}{\circ}-\cdots-\overset{-b_r}{\circ}$ *is of class* T*, then so are*

$$\overset{-2}{\circ}-\overset{-b_1}{\circ}-\cdots-\overset{-b_{r-1}}{\circ}-\overset{-b_r-1}{\circ} \quad \textit{and} \quad \overset{-b_1-1}{\circ}-\overset{-b_2}{\circ}-\cdots-\overset{-b_r}{\circ}-\overset{-2}{\circ}.$$

(3) *Every singularity of class* T *that is not a rational double point can be obtained by starting with one of the singularities described in* (1) *and iterating the steps described in* (2)*.*

Let X be a normal projective surface with singularities of class T. Due to the result of Kollár and Shepherd-Barron [8], there is a $\mathbb{Q}$-Gorenstein smoothing locally for each singularity of class T on X (see Proposition 2.5). The natural question arises whether this local $\mathbb{Q}$-Gorenstein smoothing can be extended over the global surface X or not. Roughly geometric interpretation is the following: Let $\cup_\alpha V_\alpha$ be an open covering of X such that each V_α has at most one singularity of class T. By the existence of a local $\mathbb{Q}$-Gorenstein smoothing, there is a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{V}_\alpha/\Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T^i_X = \mathcal{E}xt^i_X(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if X is a smooth surface, then T^0_X is the usual holomorphic tangent sheaf T_X and $T^1_X = T^2_X = 0$. By applying the standard result of deformations [10, 14] to a normal projective surface with quotient singularities, we get the following

Proposition 2.3 ([16], §4). *Let* X *be a normal projective surface with quotient singularities. Then*

(1) *The first order deformation space of* X *is represented by the global Ext 1-group* $\mathbb{T}^1_X = \mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X)$*.*
(2) *The obstruction lies in the global Ext 2-group* $\mathbb{T}^2_X = \mathrm{Ext}^2_X(\Omega_X, \mathcal{O}_X)$*.*

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves ([14], §3) to deformation theory of surfaces with quotient singularities so that $E^{p,q}_2 = H^p(T^q_X) \Rightarrow \mathbb{T}^{p+q}_X$, and by $H^j(T^i_X) = 0$ for $i, j \geq 1$, we also get

Proposition 2.4 ([12, 16]). *Let* X *be a normal projective surface with quotient singularities. Then*

(1) *We have the exact sequence*

$$0 \to H^1(T^0_X) \to \mathbb{T}^1_X \to \ker[H^0(T^1_X) \to H^2(T^0_X)] \to 0$$

where $H^1(T^0_X)$ *represents the first order deformations of* X *for which the singularities remain locally a product.*
(2) *If* $H^2(T^0_X) = 0$*, every local deformation of the singularities may be globalized.*

The vanishing $H^2(T^0_X) = 0$ can be obtained via the vanishing of $H^2(T_V(-\log\ E))$, where V is the minimal resolution of X and E is the reduced exceptional divisors. Note that every singularity of class T has a local $\mathbb{Q}$-Gorenstein smoothing by Proposition 2.5 below.

Let X be a normal projective surface with singularities of class T. Our concern is to understand $\mathbb{Q}$-Gorenstein smoothings in $\mathbb{T}^1_X$, not the whole first order deformations. These special deformations can be constructed via local index one cover. Let $U \subset X$ be an analytic neighborhood with an index one cover U'. For the case of the field $\mathbb{C}$,

this index one cover is unique up to isomorphism. The first order deformations which associate $\mathbb{Q}$-Gorenstein smoothings can be realized as the invariant part of $T^1_{U'}$. The sheaves $\tilde{T}^1_X$ are defined by the index one covering stack and by the étale sites [5]. The first order deformation of a $\mathbb{Q}$-Gorenstein smoothing of singularities of class T is expressed by the cohomology $H^0(\tilde{T}^1_X)$ [5, 7, 8]. By the help of the birational geometry in threefolds and their applications to deformations of surface singularities, the following proposition is obtained. Note that the cohomology $H^0(\tilde{T}^1_X)$ is given explicitly as follows.

Proposition 2.5 ([8, 12])**.** (1) *Let $a, d, n > 0$ be integers with a, n relatively prime and consider a map $\pi : \mathcal{Y}/\mu_n \to \mathbb{C}^d$, where $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^d$ is the hypersurface of equation $uv - y^{dn} = \sum_{k=0}^{d-1} t_k y^{kn}$; $t_0, \ldots, t_{d-1}$ are linear coordinates over $\mathbb{C}^d$, μ_n acts on $\mathcal{Y}$ by*

$$\mu_n \ni \xi : (u, v, y, t_0, \ldots, t_{d-1}) \to (\xi u, \xi^{-1} v, \xi^a y, t_0, \ldots, t_{d-1})$$

and π is the factorization to the quotient of the projection $\mathcal{Y} \to \mathbb{C}^d$. Then π is a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity of a germ $(X_0, 0)$ of type $\frac{1}{dn^2}(1, dna-1)$. Moreover every $\mathbb{Q}$-Gorenstein smoothing of $(X_0, 0)$ is isomorphic to the pull-back of π for some germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$.

(2) *Let X be a normal projective surface with singularities of class T. Then*

$$H^0(\tilde{T}^1_X) = \sum_{p \in \text{singular points of } X} \mathbb{C}_p^{\oplus d_p}$$

where a singular point p is of type $\frac{1}{d_p n^2}(1, d_p a n - 1)$ with $(a, n) = 1$.

Theorem 2.1 ([9])**.** *Let X be a normal projective surface with singularities of class T. Let $\pi : V \to X$ be the minimal resolution and let E be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log\ E)) = 0$. Then $H^2(T^0_X) = 0$ and there is a $\mathbb{Q}$-Gorenstein smoothing of X.*

As we see in Theorem 2.1 above, if $H^2(T^0_X) = 0$, then there is a $\mathbb{Q}$-Gorenstein smoothing of X. For example, we constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by proving the cohomology $H^2(T^0_X) = 0$ [9]. But, in general, the cohomology $H^2(T^0_X)$ is not zero and it is a very difficult problem to determine whether there exists a $\mathbb{Q}$-Gorenstein smoothing of X. Hence, in the case that $H^2(T^0_X) \neq 0$, we have to develop another technique in order to investigate the existence of $\mathbb{Q}$-Gorenstein smoothings. Even though we do not know whether such a technique exists in general, if X is a normal projective surface with singularities of class T which admits a cyclic group with some nice properties, then we are able to show that it admits a $\mathbb{Q}$-Gorenstein smoothing. Explicitly, we get the following theorem.

Theorem 2.2. *Let X be a normal projective surface with singularities of class T. Assume that a cyclic group G acts on X such that*

(1) *$Y = X/G$ is a normal projective surface with singularities of T,*
(2) *$p_g(Y) = q(Y) = 0$,*
(3) *Y has a $\mathbb{Q}$-Gorenstein smoothing,*
(4) *the map $\sigma : X \to Y$ induced by a cyclic covering is flat, and the branch locus D (resp. the ramification locus) of the map $\sigma : X \to Y$ is an irreducible nonsingular curve lying outside the singular locus of Y (resp. of X), and*
(5) *$H^1(Y, \mathcal{O}_Y(D)) = 0$.*

Then there exists a $\mathbb{Q}$-Gorenstein smoothing of X that is compatible with a $\mathbb{Q}$-Gorenstein smoothing of Y. And the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing.

Proof. Let $\mathcal{Y} \to \Delta$ be a $\mathbb{Q}$-Gorenstein smoothing of Y, and let Y_t be a general fiber of the $\mathbb{Q}$-Gorenstein smoothing. By the semi-continuity, we have $p_g(Y_t) = q(Y_t) = 0$. The base change theorem and Leray spectral sequence imply that $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = H^2(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 0$. It gives an isomorphism $r_0 : \mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(Y)$ and an injective map $r_t : \mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(Y_t)$ (Lemma 2 in [12]). The vanishing $H^1(Y, \mathcal{O}_Y(D)) = 0$ ensures that the deformation of Y can be lifted to the deformation of the pair (Y, D), i.e. the branch divisor D is extended to D_t in Y_t. Since the divisor D is nonsingular, D_t is also nonsingular. And the flatness of the map ensures that the divisor L which is the data of the cyclic cover, i.e. $L^{\otimes|G|} \cong D$, is extended to L_t with $L_t^{\otimes|G|} \cong D_t$. Hence, the cyclic covering extends to the $\mathbb{Q}$-Gorenstein smoothing of Y. $\square$

3. A construction of Horikawa surface

Let $E(n)$ be a simply connected relatively minimal elliptic surface with a section and with $c_2 = 12n$. Then there is only one up to diffeomorphism such an elliptic surface and the canonical class is given by $K_{E(n)} = (n-2)C$, where C is a general fiber of an elliptic fibration. Hence each section is a nonsingular rational curve whose self-intersection number is $-n$. Assume that $n \geq 4$ and let C_{n-2} denote a simply connected smooth 4-manifold obtained by pluming the $(n-3)$ disk bundles over the 2-sphere according to the linear diagram

$$\underset{U_{n-3}}{\overset{-n}{\circ}} - \underset{U_{n-4}}{\overset{-2}{\circ}} - \underset{U_{n-5}}{\overset{-2}{\circ}} - \cdots - \underset{U_1}{\overset{-2}{\circ}}.$$

Assume that an elliptic surface $E(n)$ has two configurations C_{n-2} such that all embedded 2-spheres U_i are holomorphic curves (We show the existence of such an $E(n)$ later). Let Y_n' be a normal projective surface obtained by contracting one configuration C_{n-2} from $E(n)$. Then Y_n' does not admit a $\mathbb{Q}$-Gorenstein smoothing because it violates Noether inequality (Corollary 7.5 in [3]). In fact, it does not satisfy the vanishing condition in the hypothesis of Theorem 2.1, that is, we have $H^2(E(n), T_{E(n)}) \neq 0$: Let $h : E(n) \to \mathbb{P}^1$ be an elliptic fibration. Assume that C is a general fiber of the map h. We have an injective map $0 \to h^*\Omega_{\mathbb{P}^1} \to \Omega_{E(n)}$ and the map induces an injection $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(n-2)) \hookrightarrow H^0(E(n), \Omega_{E(n)}((n-2)C))$ by tensoring $(n-2)C$ on $0 \to h^*\Omega_{\mathbb{P}^1} \to \Omega_{E(n)}$. Since $K_{E(n)} = (n-2)C$, the cohomology $H^0(E(n), \Omega_{E(n)}(K_{E(n)}))$ is not zero. Hence the Serre duality implies that $H^2(E(n), T_{E(n)})$ is not zero.

Next, let X_n be a normal projective surface obtained by contracting two disjoint configurations C_{n-2} from $E(n)$, and we want to investigate the existence of a $\mathbb{Q}$-Gorenstein smoothing of X_n. As a warming-up, we first investigate $n = 4$ case.

Example. R. Gompf constructed a family of symplectic 4-manifolds by taking a fiber sum of other symplectic 4-manifolds [4]. To recall Gompf's example briefly, we start with a simply connected relatively minimal elliptic surface $E(4)$ with a section and with $c_2 = 48$. It is known that $E(4)$ admits nine rational (-4)-curves as disjoint sections. Rationally blowing-down n (-4)-curves of $E(4)$ is the same as the normal connected sum of $E(4)$ with n copies of $\mathbb{P}^2$ by identifying a conic in each $\mathbb{P}^2$ with one (-4)-curve in $E(4)$. Let us denote this 4-manifold by $W_{4,n}$. Then the manifold $W_{4,1}$ does not admit

any complex structure because it violates the Noether inequality. But we will show that $W_{4,2}$ admits a complex structure using a $\mathbb{Q}$-Gorenstein smoothing theory. For this, let us first denote the singular projective surface obtained by contracting n (-4)-sections from $E(4)$ by $W'_{4,n}$. And then we claim that $W'_{4,2}$ has a $\mathbb{Q}$-Gorenstein smoothing. The reason is following: Consider $E(4)$ as a double cover of Hirzebruch surface $\mathbb{F}_4$ branched over an irreducible nonsingular curve D in the linear system $|4(C_0+4f)|$, where C_0 is a negative section and f is a fiber of $\mathbb{F}_4$. Then $H^1(\mathbb{F}_4, \mathcal{O}_{\mathbb{F}_4}(D)) = 0$: Since $p_g(\mathbb{F}_4) = q(\mathbb{F}_4) = 0$,

$$H^1(\mathbb{F}_4, \mathcal{O}_{\mathbb{F}_4}(D)) \simeq H^1(D, \mathcal{O}_D(D)) \simeq H^0(D, \mathcal{O}_D(K_D - D))^\vee.$$

And $\deg K_D - D^2 = 4(C_0+4f)(2C_0+10f) - 16(C_0+4f)^2 = -24 < 0$ implies that $H^0(D, \mathcal{O}_D(K_D - D)) = 0$. Since D does not intersect C_0, $W'_{4,2}$ is a double cover of a cone $\hat{\mathbb{F}}_4$ which is a contraction of C_0 from $\mathbb{F}_4$. This implies that the map σ induced by a double cover is flat and $H^1(Y, \mathcal{O}_Y(D)) = 0$. Note that $\hat{\mathbb{F}}_4$ has a $\mathbb{Q}$-Gorenstein smoothing whose general fiber is $\mathbb{P}^2$. It is obtained by a pencil of hyperplane section of the cone of the Veronese surface imbedded in $\mathbb{P}^5$. Hence $W'_{4,2}$ has a $\mathbb{Q}$-Gorenstein smoothing by Theorem 2.2. Finally, since the rational blow-down manifold $W_{4,2}$ is diffeomorphic to the general fiber of the $\mathbb{Q}$-Gorenstein smoothing of $W'_{4,2}$, $W_{4,2}$ admits a complex structure. Furthermore, using a triple cover of $\mathbb{F}_4$ branched over D in the linear system $|3(C_0+4f)|$, we can also prove that $W'_{4,3}$ has a $\mathbb{Q}$-Gorenstein smoothing by the similar proof as above. And, by extending Theorem 2.2 to a finite abelian group, it is possible to show that some other manifolds $W'_{4,n}$ has a $\mathbb{Q}$-Gorenstein smoothing, too. We leave it for a future research.

Now we investigate the general case. Assume that $n \geq 5$ and let $\mathbb{F}_n$ be a Hirzebruch surface. Let C_0 be a negative section with $C_0^2 = -n$ and f be a fiber of $\mathbb{F}_n$. Consider the linear system $|4(C_0+nf)|$. The surface $\mathbb{F}_n$ can be obtained from the cone over a rational normal curve of degree n by blowing up the vertex. And a curve in the linear system $|4(C_0+nf)|$ is the strict transform of the hyperplane section of the cone. By Bertini's theorem, there is an irreducible nonsingular curve in the linear system $|4(C_0+nf)|$. The double cover of $\mathbb{F}_n$ branched over an irreducible nonsingular member in $|4(C_0+nf)|$ is an elliptic surface $E(n)$: Let $\sigma : \hat{X}_n \to \mathbb{F}_n$ be a double covering branched over an irreducible nonsingular member in the linear system $|4(C_0+nf)|$. Then, by the invariants of a double covering ([2], Chapter V), we have $p_g(\hat{X}_n) = p_g(\mathbb{F}_n) + h^0(\mathbb{F}_n, K_{\mathbb{F}_n} + L) = h^0(\mathbb{F}_n, (n-2)f) = n-1$ and $\chi(\mathcal{O}_{\hat{X}_n}) = 2\chi(\mathcal{O}_{\mathbb{F}_n}) + \frac{1}{2}(L\cdot K_{\mathbb{F}_n}) + \frac{1}{2}(L\cdot L) = n$, where $L = 2(C_0+nf)$. Therefore we have $q(\hat{X}_n) = 0$ and $K^2_{\hat{X}_n} = 2(\sigma^*(K_{\mathbb{F}_n}+L))^2 = 2((n-2)f)^2 = 0$.

In this article we want to choose a special irreducible (singular) curve D in the linear system $|4(C_0+nf)|$, which has a special intersection with one special fiber f: Note that $D\cdot f = 4$. We want D to intersect with f at two distinct points p and q that are not in C_0. Let $x = 0$ be the local equation of f and x, y be a coordinate at p (resp. at q). We require that the local equation of D at p (resp. at q) is $(y-x)(y+x) = 0$ (resp. $(y-x^{n-4})(y+x^{n-4}) = 0$). These are $3(n-4)+3$-conditions: $1, x, x^2, \ldots, x^{2n-9}, y, yx, \ldots, yx^{n-5}$ terms should vanish to have the local analytic equation $(y-x^{n-4})(y+x^{n-4}) = 0$. By next lemmas and proposition, we have such a curve D satisfying the conditions above.

Lemma 3.1. *We have $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(D)) = 10n+5$, where D is a member in the linear system $|4(C_0+nf)|$.*

Proof. Let $C = C_0 + nf$. Then, by the following two exact sequences

$$0 \to \mathcal{O}_{\mathbb{F}_n} \to \mathcal{O}_{\mathbb{F}_n}(f) \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

$$0 \to \mathcal{O}_{\mathbb{F}_n}(kf) \to \mathcal{O}_{\mathbb{F}_n}((k+1)f) \to \mathcal{O}_{\mathbb{P}^1} \to 0,$$

we have $h^1(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(kf)) = 0$ for all nonnegative integers k. And from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(nf) \to \mathcal{O}_{\mathbb{F}_n}(C) \to \mathcal{O}_{C_0}(C) \to 0,$$

we also have $h^1(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(C)) = 0$ and $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(C)) = n + 2$. Hence, by considering the exact sequences similarly

$$0 \to \mathcal{O}_{\mathbb{F}_n}(kC) \to \mathcal{O}_{\mathbb{F}_n}((k+1)C) \to \mathcal{O}_C((k+1)C) \to 0,$$

we finally get $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(D)) = 10n + 5$. $\square$

First, we assume that D is nonsingular at every point except the two points p and q. Let $\sigma : \tilde{X}_n \to \mathbb{F}_n$ be a double covering branched over the curve D chosen above. Then $\tilde{X}_n$ is a singular elliptic surface with $p_g = n - 1$ and $\chi = n$ which has two rational double points by the local equations of D at p and q - one is A_1 ($z^2 = y^2 - x^2$) and the other one is A_{2n-9} ($z^2 = y^2 - x^{2n-8}$). Therefore its minimal resolution is also an elliptic surface $E(n)$. First we blow up at p and q in $\mathbb{F}_n$. Then we have an exceptional curve coming from a blowing up at p which intersects with the proper transform of D transversally at two points, and we also have an exceptional curve coming from a blowing up at q which intersects with the proper transform of D at one point, say q_1. Let $x = 0$ be the local equation of the (-1)-exceptional curve at q_1. Then the local equation of the proper transform of D at q_1 is $(y - x^{n-5})(y + x^{n-5}) = 0$. We blow up again at q_1. By the continuation of blowing up at infinitely near points of q, we have the following configuration of smooth rational curves

$$\begin{array}{ccccccccc}
\overset{-n}{\circ} & - & \overset{-2}{\circ} & - & \overset{-2}{\circ} & - & \cdots & - & \overset{-2}{\circ} \\
U_{n-3} & & U_{n-4} & & U_{n-5} & & & & U_1 \\
 & & | & & & & & & | \\
 & & \circ & & & & & & \circ \\
 & & E_1, -1 & & & & & & E_2, -1
\end{array}$$

where the proper transform of D intersects with E_i, $i = 1, 2$ at two points transversally. We denote this surface by Z_n, which is obtained by $(n-3)$ times blowing-ups of $\mathbb{F}_n$.

Let $\pi : Z_n \to \mathbb{F}_n$ be a map and $\Delta = \pi^*(4C_0 + 4nf) - 2E_1 - 2U_{n-5} - 4U_{n-6} - 6U_{n-7} - \cdots - 2(n-5)U_1 - 2(n-4)E_2$. For a simple computation, we write it as

$$\Delta = \pi^*(4C_0 + 4nf) - 2F - 2\sum_{i=1}^{n-4} F_i,$$

where $F = E_1$, $F_1 = U_{n-5} + U_{n-6} + \cdots + E_2$, $F_2 = U_{n-6} + \cdots + E_2, \ldots, F_{n-4} = E_2$. Note that F_i is not necessarily irreducible and $F^2 = F_i^2 = K_{Z_n} \cdot F = K_{Z_n} \cdot F_i = -1$ for all $i = 1, \ldots, n-4$. Let $f_0 = U_{n-4}$, which is a proper transform of the fiber, and let $L = \Delta - (\pi^* C_0 + f_0) - K_{Z_n}$.

In Proposition 3.1 below, we prove that the linear system $|\Delta|$ is base point free. Then, by Bertini's theorem, we conclude that D is nonsingular except the two points p and q.

Lemma 3.2. *$L^2 \geq 5$ and L is nef.*

Proof. Since $\pi^* f = f_0 + F + \sum_{i=1}^{n-4} F_i$, we have $L = \pi^*(5C_0 + (5n+2)f) - f_0 - 3F - 3\sum_{i=1}^{n-4} F_i = \pi^*(5C_0 + (5n-1)f) + 2f_0$. Furthermore, since the two points p and q are not in C_0, we also have $C_0 \cdot f = C_0 \cdot f_0 = 1$ and $f_0^2 = -2$. Therefore

$$L^2 = -25n + (5n-1)10 + 20 - 8 = 25n + 2 \geq 5.$$

Let G be an irreducible curve which is neither f_0 nor C_0. Note that $L \cdot f_0 = 5 - 4 = 1$ and $L \cdot C_0 = -5n + 5n - 1 + 2 = 1$. Write $\pi_* G = aC_0 + bf$. Then we have

$$G \cdot L \geq \pi_* G \cdot (5C_0 + (5n-1)f) = (aC_0 + bf) \cdot (5C_0 + (5n-1)f) = -a + 5b.$$

We note that the linear system $|aC_0 + bf|$ contains an irreducible curve in $\mathbb{F}_n$ if and only if $a = 0, b = 1$; or $a = 1, b = 0$; or $a > 0, b > an$; or $a > 0, b = an$ with $n > 0$ (refer to Corollary 2.18, Chapter V in [6]). Therefore it is impossible that $G \cdot L < 0$. It implies that L is nef. □

Lemma 3.3. *The linear system $|\Delta - (\pi^* C_0 + f_0)|$ on Z_n is base point free.*

Proof. By Lemma 3.2 above, L is nef and $L^2 \geq 5$. Hence, applying to Reider's theorem [15], if the adjoint linear series $|\Delta - (\pi^* C_0 + f_0)| = |K_{Z_n} + L|$ has a base point at x then there is an effective divisor G in Z_n passing through x such that either $G \cdot L = 0$ and $G^2 = -1$; or $G \cdot L = 1$ and $G^2 = 0$.

Assume that $G \cdot L = 0$ and $G^2 = -1$. Write $G = G_1 + \cdots + G_k$, where G_k is an irreducible curve. Since L is nef, $G \cdot L = 0$ implies that $G_i \cdot L = 0$ for all $i = 1, \ldots, k$. Then we get a contradiction by a similar argument to show that $-a + 5b \leq 0$ in the proof Lemma 3.2 above.

Assume that $G \cdot L = 1$ and $G^2 = 0$. By the same argument in the case $G \cdot L = 0$, $G = G_1$. Then we get a contradiction by a similar argument to show that $-a + 5b \leq 1$ unless $G = C_0$ or f_0. Furthermore, since $C_0^2 = -n$ and $f_0^2 = -2$, it also contradicts. □

Proposition 3.1. *The linear system $|\Delta|$ on Z_n is base point free.*

Proof. Note that $\Delta \cdot \pi^* C_0 = \Delta \cdot f_0 = 0$. Therefore we have $\mathcal{O}_{\pi^* C_0 + f_0}(\Delta) = \mathcal{O}_{\pi^* C_0 + f_0}$. Furthermore, by Lemma 3.2 above and by the vanishing theorem, we also have

$$H^1(Z_n, \Delta - (\pi^* C_0 + f_0)) = H^1(Z_n, K_{Z_n} + L) = 0.$$

Hence, using Lemma 3.3 above and the short exact sequence

$$0 \to \mathcal{O}_{Z_n}(\Delta - (\pi^* C_0 + f_0)) \to \mathcal{O}_{Z_n}(\Delta) \to \mathcal{O}_{\pi^* C_0 + f_0} \to 0,$$

we conclude that the linear system $|\Delta|$ is base point free. □

Next, by Artin's criterion of contraction [1], we can contract a configuration C_{n-2}, which is a linear chain of $\mathbb{P}^1$'s

$$\underset{U_{n-3}}{\overset{-n}{\circ}} - \underset{U_{n-4}}{\overset{-2}{\circ}} - \underset{U_{n-5}}{\overset{-2}{\circ}} - \cdots - \underset{U_1}{\overset{-2}{\circ}},$$

so that it produces a singular normal projective surface. We denote this surface by Y_n. We note that Δ is the proper transform of D in Z_n and that Y_n has a cyclic quotient singularity of type $\frac{1}{(n-2)^2}(1, n-3)$, which is a singularity of class T.

Lemma 3.4. $H^1(Z_n, \mathcal{O}_{Z_n}(\Delta)) = 0$.

Proof. Since $p_g(Z_n) = q(Z_n) = 0$, we have

$$H^1(Z_n, \mathcal{O}_{Z_n}(\Delta)) \simeq H^1(\Delta, \mathcal{O}_\Delta(\Delta)) \simeq H^0(\Delta, \mathcal{O}_\Delta(K_\Delta - \Delta))^\vee.$$

We also have $\Delta^2 = D^2 - 4(n-3) = 12n + 12$ and $\deg K_\Delta = \deg K_D - 2(n-3) = 12n - 8 - 2(n-3) = 10n - 2$. Therefore it satisfies

$$\deg K_\Delta - \Delta^2 = 10n - 2 - 12n - 12 < 0,$$

and it implies that $H^0(\Delta, \mathcal{O}_\Delta(K_\Delta - \Delta)) = 0$. □

Proposition 3.2. *The singular surface Y_n admits a $\mathbb{Q}$-Gorenstein smoothing.*

Proof. It is enough to show that $-K_{Y_n}$ is effective (Theorem 21 in [12]). Let $\pi : Z_n \to \mathbb{F}_n$ be a composition of blowing-ups, and $\psi : Z_n \to Y_n$ be a contraction. Then we have

$$K_{Z_n} = \pi^* K_{\mathbb{F}_n} + E_1 + U_{n-5} + 2U_{n-6} + \cdots + (n-5)U_1 + (n-4)E_2.$$

Since $K_{\mathbb{F}_n} = -2C_0 - (n+2)f$, $-K_{Z_n}$ is effective. Furthermore, since $h^0(-K_{Y_n}) = h^0(\psi_*(-K_{Z_n})) = h^0(-K_{Z_n})$ (§3.9.2 in [16]), $-K_{Y_n}$ is also effective. □

Now we are in a position to prove our main theorem mentioned in the Introduction. First remind that Horikawa surface $H(n)$ is a double cover of $\mathbb{F}_{n-3}$ branched over a smooth curve D_{n-2} in the linear system $|6C_0 + (4n-8)f|$. R. Fintushel and R. Stern showed that Horikawa surface can be decomposed into

$$H(n) = B_{n-2} \cup D_{n-2} \cup B_{n-2},$$

where B_{n-2} is the complement of a neighborhood of the pair of 2-spheres $(C_0 + (n-2)f)$ and C_0 in $\mathbb{F}_{n-3}$ (Lemma 2.1 in [3]), and they proved that an elliptic surface $E(n)$ is obtained from $H(n)$ by replacing two rational balls B_{n-2} with two configurations C_{n-2} (Lemma 7.3 in [3]). In other words, R. Fintushel and R. Stern proved that Horikawa surface $H(n)$ can be obtained from an elliptic surface $E(n)$ by rationally blowing-down two disjoint configurations C_{n-2} lying in $E(n)$ in smooth category. The aim of this article is to prove that the rational blow-down surgery above can be performed in complex category, which is following.

Proof of Theorem 1.1. Note that $\tilde{X}_n$ is a double covering of $\mathbb{F}_n$ branched over D, and the minimal resolution of two rational double points of type A_1 and A_{2n-9} in $\tilde{X}_n$ is $E(n)$, which is also a double cover of Z_n branched over the proper transform of D. Since the proper transform of D does not meet the contracted linear chain of $\mathbb{P}^1$'s, we have a double cover of Y_n branched over the image of the proper transform of D by the map ψ. We denote this surface by X_n. Then X_n is the singular surface obtained by contracting two disjoint configurations C_{n-2} from an elliptic surface $E(n)$ and it has two quotient singularities of class T, both are of type $\frac{1}{(n-2)^2}(1, n-3)$. By the fact that the proper transform of D is disjoint from the contracted liner chain of $\mathbb{P}^1$'s and Lemma 3.4, the map from X_n to Y_n is flat and $H^1(Y_n, \mathcal{O}_{Y_n}(\bar{D}_{Y_n})) = 0$, where $\bar{D}_{Y_n}$ is the image of Δ in Y_n under the contraction C_{n-2} of the liner chain of $\mathbb{P}^1$'s. Therefore we have the following commutative diagram of maps

$$\begin{array}{ccccc} \tilde{X}_n & \leftarrow & E(n) & \rightarrow & X_n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{F}_n & \stackrel{\pi}{\leftarrow} & Z_n & \stackrel{\psi}{\rightarrow} & Y_n \end{array}$$

where all vertical maps are double coverings. Then, by Theorem 2.2 and Proposition 3.2 above, the singular surface X_n has a $\mathbb{Q}$-Gorenstein smoothing of two quotient singularities all together.

Finally, by applying the standard arguments about Milnor fibers (§5 in [11] or §1 in [13]), we know that a general fiber of a $\mathbb{Q}$-Gorenstein smoothing of X_n is diffeomorphic to the 4-manifold obtained by rational blow-down of $E(n)$. And we also know that $H(n)$ has one deformation class ([2], Chapter VII). Therefore a general fiber of a $\mathbb{Q}$-Gorenstein smoothing of X_n is a Horikawa surface $H(n)$ in complex category. □

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